Name: ____________________

Class: 12MTZ1

Teacher: MR TONG

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2013 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Multiple choice questions are to be answered on the multiple choice sheet provided at the back of the paper.
- Each question in Section 2 is to be commenced in a new booklet clearly marked Question 11, Question 12, etc on the top of the page and must show your name and class.
- All necessary working should be shown in every question in Section 2. Full marks may not be awarded for careless or badly arranged work.
- Board of Studies approved calculators may be used. Standard Integral Tables are provided.
- Write your name and class in the space provided at the top of this question paper.

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

---

1 Let $z = 4 - i$. What is the value of $\overline{iz}$?

(A) $-1-4i$ (B) $-1+4i$

(C) $1-4i$ (D) $1+4i$

2 The equation $2x^3 - 7x + 1 = 0$ has root α, β and γ. What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) 0 (B) $\frac{43}{4}$

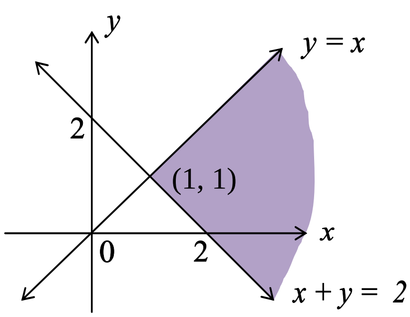
(C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

3. The complex number z satisfies the inequations

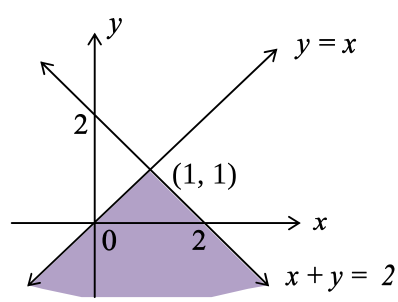
$|z + 2i| \geq |z + 2|$ and $\text{Im}(z) + \text{Re}(z) \geq 2$

Which of these shows the shaded region in the Argand diagram that satisfies these inequations?

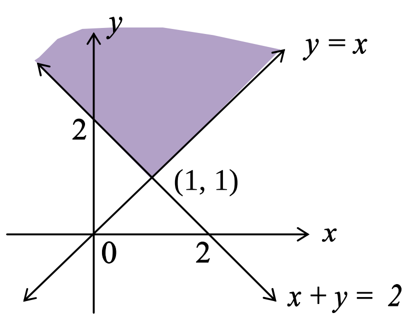
(A)



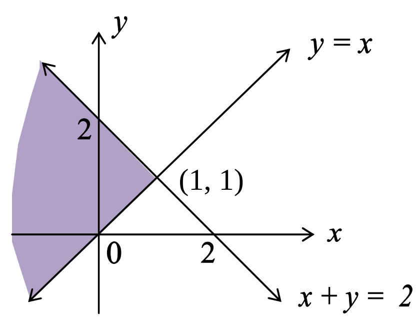
(B)



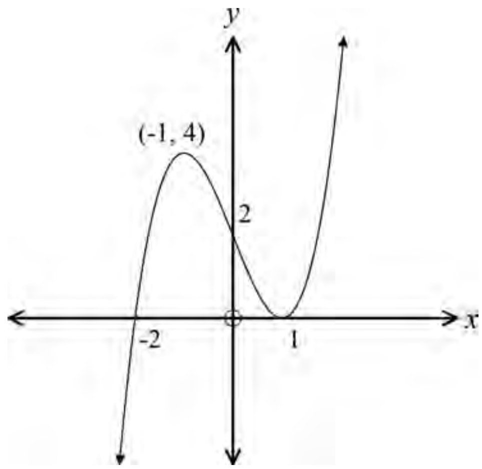
(C)



(D)

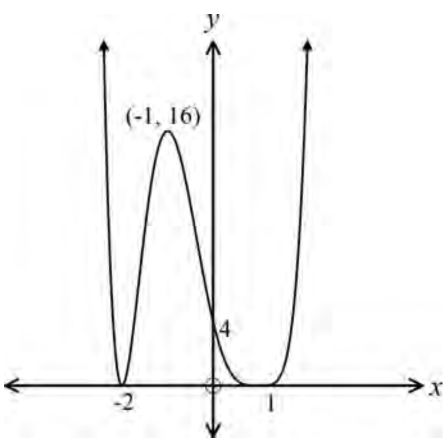


4. The graph of the function $y = f(x)$ is drawn below.

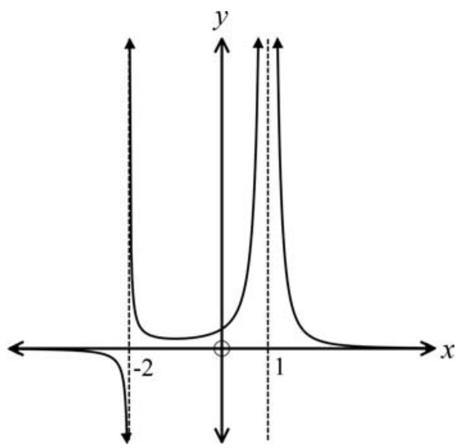


Which of the following graphs best represents the graph $y = \sqrt{f(x)}$?

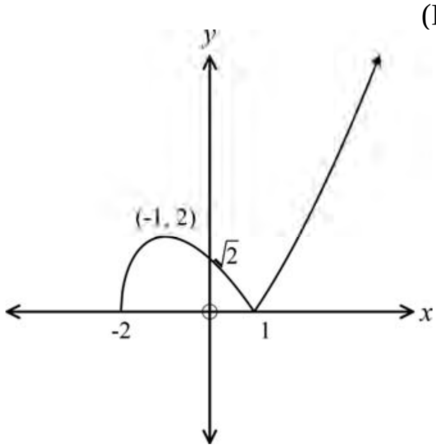
(A)



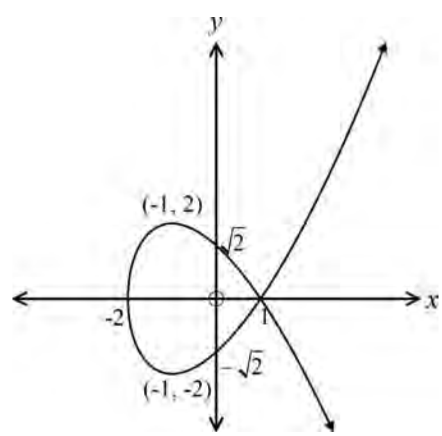
(B)



(C)



(D)



5. Consider the equation $2x^3 - 3x^2 + 2x + 2 = 0$.

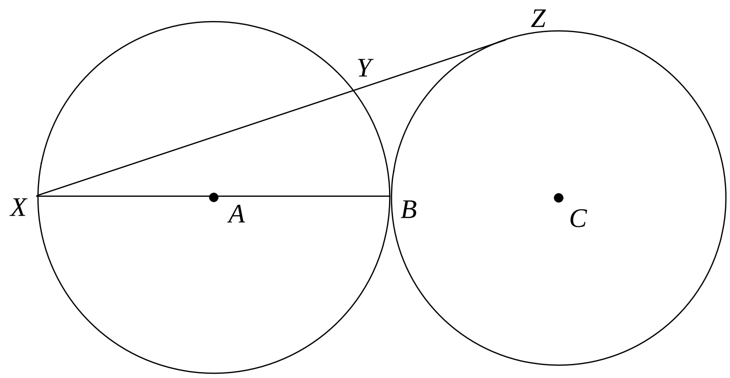
The roots of this equation are α, β and γ.

What is the value of $\alpha + \beta - \dfrac{1}{\alpha\beta}$?

(A) $\dfrac{3}{2}$ (B) $-\dfrac{3}{2}$

(C) 1 (D) -1

6. Two equal circles touch externally at B. XB is a diameter of one circle. XZ is the tangent from X to the other circle and cuts the first circle at Y.



Which is the correct expression that relates XZ to XY?

(A) $3XZ = 4XY$

(B) $2XZ = 3XY$

(C) $XZ = 2XY$

(D) $2XZ = 5XY$

7. Which integral has the smallest value?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \; dx$

(B) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \; dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x \, dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \tan x \; dx$

8. What is the derivative of $\cos^{-1} x - \sqrt{1-x^2}$?

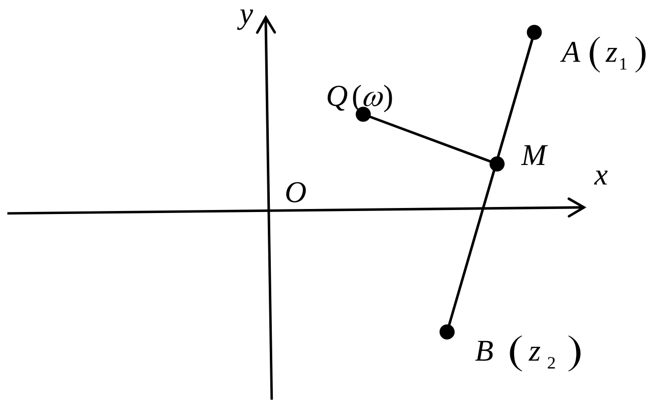
(A) $-\sqrt{\dfrac{1-x}{1+x}}$

(B) $-\dfrac{\sqrt{1-x}}{1+x}$

(C) $\dfrac{x-1}{\sqrt{1+x}}$

(D) $\dfrac{x-1}{x+1}$

9. On the Argand diagram below, points A and B correspond to the complex numbers z_1 and z_2 respectively. M is the mid-point of the interval AB and QM is drawn perpendicular to AB. $QM = AM = BM$.
If Q corresponds to the complex number ω then $\omega = ?$



(A) $i\left(\dfrac{z_1 - z_2}{2}\right)$

(B) $i\left(\dfrac{z_1 + z_2}{2}\right)$

(C) $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 + z_2}{2}\right)$

(D) $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 - z_2}{2}\right)$

10. A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk\left(v + v^2\right)$ newtons when its speed is $v\ ms^{-1}$ (where k is a positive constant). At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity $v\ ms^{-1}$. Which of the following is an expression for x in terms of v?

(A) $\frac{1}{k}\int \frac{1}{1+v}\,dv$

(B) $\frac{1}{k}\int \frac{1}{v(1+v)}\,dv$

(C) $-\frac{1}{k}\int \frac{1}{v(1+v)}\,dv$

(D) $-\frac{1}{k}\int \frac{1}{1+v}\,dv$

Section II

Total marks (90)
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer all questions, starting each question in a new writing booklet. with your name and question number on the front of the booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet

Marks

a) Let $z = 1 + \sqrt{3}\, i$.

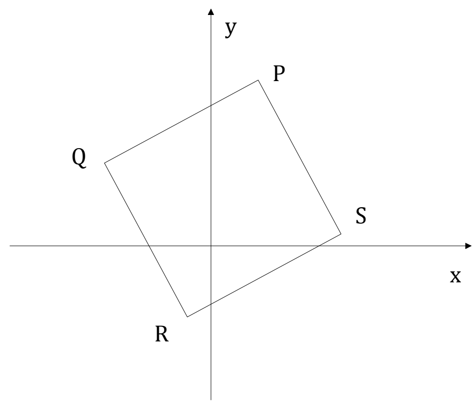
(i) Find the exact values of $|z|$ and $\arg z$. **2**

(ii) Find the exact value of z^5 in the form of $a + bi$ where a, b are real numbers. **2**

(iii) List the complex 4^{th} roots of z. Leave your answer in mod-arg form. **2**

b) Find the square roots of 15 − 8i. **2**

c) In the Argand diagram below, vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$ represents the complex numbers p, q, r and s respectively, where PQRS is a square. **2**



Show that $s + ip = q + ir$.

d) (i) Find real numbers a, b, c and d such that $\dfrac{2x^3 - 9x^2 + 18x - 9}{\left(1 + x^2\right)\left(9 + x^2\right)} = \dfrac{ax + b}{1 + x^2} + \dfrac{cx + d}{9 + x^2}$ **3**

(ii) Hence, evaluate in simplest form **2**

$$\int_0^3 \frac{2x^3 - 9x^2 + 18x - 9}{\left(1 + x^2\right)\left(9 + x^2\right)}\, dx \ .$$

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

a) (i) Show that if $x = a$ is a double root of the polynomial $P(x) = 0$, then $P'(a) = P(a) = 0$. **2**

(ii) Find the roots of the equation $x^4 - 2x^3 + x^2 + 12x + 8 = 0$, given that the equation has a double root **3**

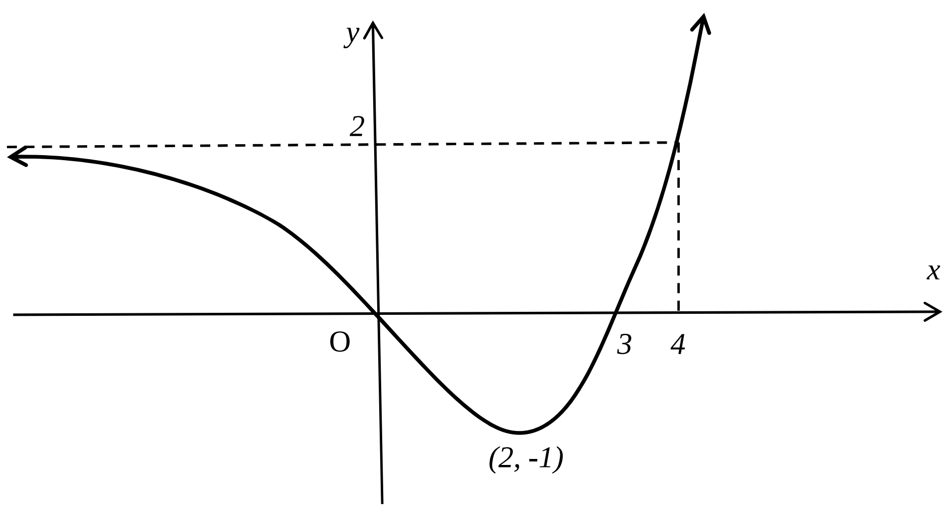
(iii) Given that one root of the equation $x^4 - 5x^3 + 5x^2 + 25x - 26 = 0$ is $3 + 2i$, solve the equation. **3**

b) The equation $2x^3 - x^2 + 3x - 1 = 0$ has roots α, β, and γ. Find the cubic equation which has roots:

(i) $\dfrac{1}{\alpha\beta}$, $\dfrac{1}{\beta\gamma}$ and $\dfrac{1}{\gamma\alpha}$. **2**

(ii) α^2, β^2 and γ^2. **2**

c) The diagram below shows the graph of $y = f(x)$.



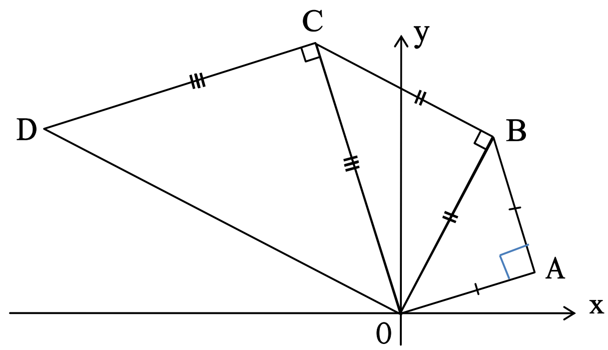
Detach the last page of the question booklet and draw separate diagrams of the following graphs. Carefully show any horizontal or vertical asymptotes and any intercepts with the coordinate axes. Please attach your solution to your writing booklet.

(i) $y = |f(x)|$ **1**

(ii) $y = [f(x)]^2$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

a)



The points A and D in a complex plane represent the complex numbers α and β respectively. The triangles OAB, OBC and OCD are right angled isosceles triangles as shown.

(i) Show that B represents the complex number $\left(\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)\right)\alpha$ **1**

(ii) Hence show that $\beta = 2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right) \times i\,\alpha$ **2**

(iii) Show that $64\alpha^4 + \beta^4 = 0$ **1**

b) Evaluate $\int_0^3 x\sqrt{x+1}\,dx$ by using a suitable substitution. **3**

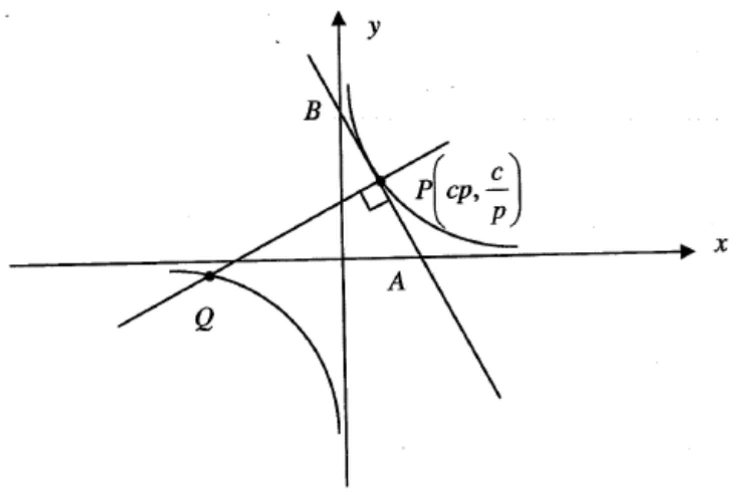
Question 13 continues on the next page

Marks

Question 13 continued

c) (i) Show that $p^2 + \frac{1}{p^2} \geq 2$. **1**

(ii) The point $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$. The tangent to the hyperbola at P cuts the x and y axes at A and B respectively and the normal to the hyperbola at P cuts the hyperbola again at Q.
The tangent at P has equation $x + p^2 y = 2cp$.



(α) Show that the length of the interval AB is $2c\sqrt{p^2 + \frac{1}{p^2}}$ units. **2**

(β) Given that the equation of the normal at P is $py - c = p^3(x - cp)$, find the coordinates of Q. **2**

(γ) Show that the area of triangle $ABQ = c^2\left(p^2 + \frac{1}{p^2}\right)^2$ units^2 **2**

(δ) Find the minimum area of triangle ABQ. [Hint: use the result of part c(i)]. **1**

Question 14 begins on the next page.

Marks

Question 14 (15 marks) Use a SEPARATE writing booklet

a) An ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

(i) Show that this is the equation of the locus of a point $P(x, y)$ that moves such that the sum of its distances from $A(0, 3)$ and $B(0, -3)$ is 10 units. **4**

(ii) Find the equation of the tangent to the ellipse at the point in the first quadrant where $y = 4$. **3**

b) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x - \sin x}\,dx$$

c) Consider the polynomial $P(x) = ax^3 + 3x + b$ where a and b are real. It has roots $m + in$, $m - in$ and $\dfrac{1}{a}$ where m and n are real and non-zero. It is known that the graph of $y = P(x)$ has two turning points.

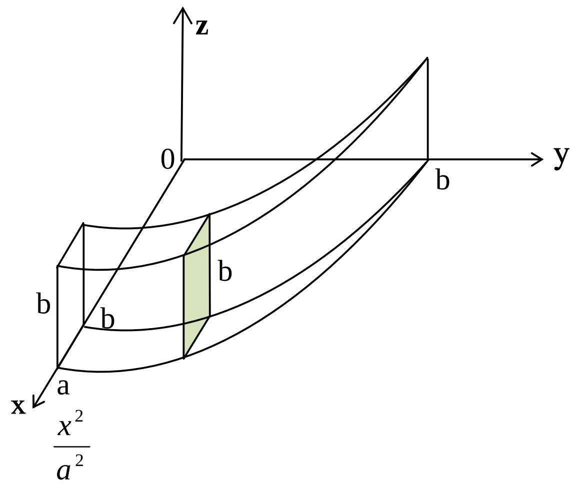
(i) Considering $P'(x)$, show that $a < 0$. **1**

(ii) Hence or otherwise, show $b < 0$. **2**

(iii) Show that $m > \dfrac{3}{2}$. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The base of the solid shown in the diagram is the region in the first quadrant bounded by the x and the y axes, the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=b^2$. Each cross section of this solid perpendicular to the y axis is a rectangle b metres high. A typical cross section is shaded. **3**



Show that the volume of the solid is given by $V = \int_0^b (a-b)\sqrt{b^2-y^2}\,dy$ and hence find the volume of the solid.

b) A particle of unit mass is moving in a straight line. It is initially at the origin and is travelling with velocity $\sqrt{3}\,ms^{-1}$. The particle is moving against a resistance $v+v^3$, where v is the velocity.

(i) Briefly explain why the acceleration of the particle is given by $a=-\left(v+v^3\right)$. **1**

(ii) Show that the displacement x of the particle from the origin is given by **4**

$$x=\tan^{-1}\left(\frac{\sqrt{3}-v}{1+v\sqrt{3}}\right).$$

(iii) Show that the time t which has elapsed when the particle is travelling with velocity V is given by $t=\frac{1}{2}\log_e\left[\frac{3\left(1+V^2\right)}{4V^2}\right]$ **4**

Question 15 continues on the next page.

Marks

Question 15 continued.

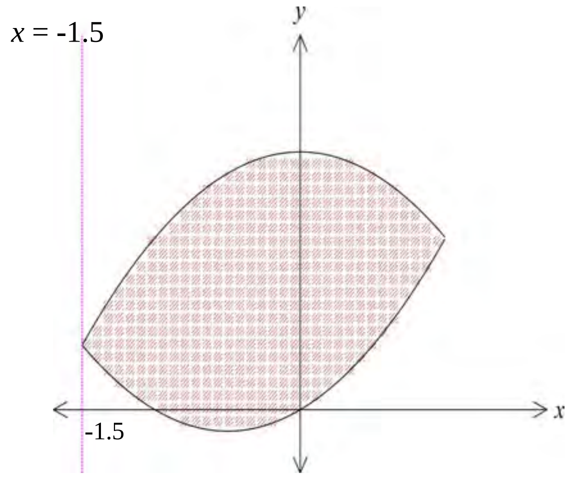
(iv) Find V^2 as a function of *t*. **2**

(v) Find the limiting position of the particle as $t \to \infty$. **1**

Question 16 begins on the next page.

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

a) A solid is formed by rotating the shaded region bounded by the curves $y = x^2 + x$ and $y = 3 - x^2$ about the line $x = -1.5$. **4**



Find the volume of this solid using the method of cylindrical shells.

b) (i) Show that $\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$. **1**

(ii) Hence, show that **3**

$$\cos 3\theta + \cos\theta + 4\cos^3\theta = 8\cos\theta\cos\left(\theta + \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right)$$

c) (i) Show that $y = x - 1$ is a tangent to the curve $y = \log_e x$ at the point where $x = 1$. **1**

(ii) Hence, or otherwise, show that $\log_e x \le x - 1$ for $x > 0$. **2**

(iii) Given n positive numbers $a_1, a_2, a_3, ..., a_n$ such that **2**

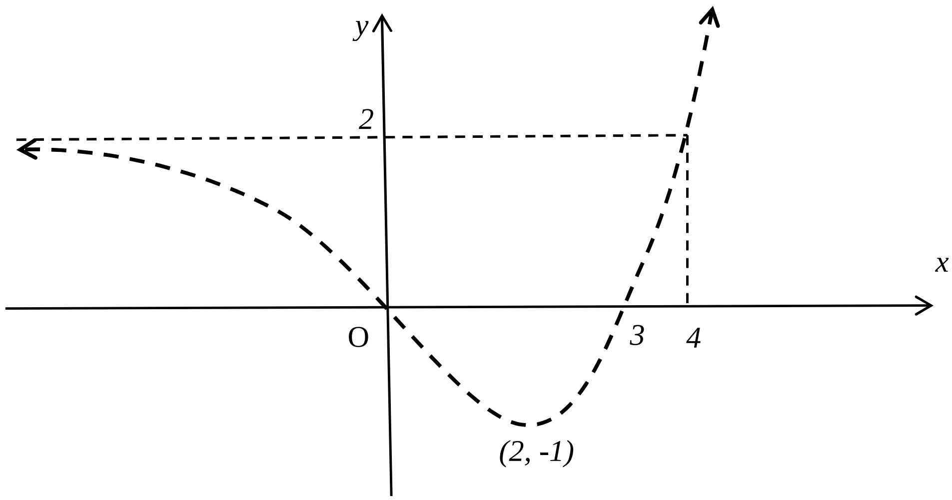
$a_1 + a_2 + a_3 + ... + a_n = 1$, prove that $\displaystyle\sum_{k=1}^{n} \log_e(na_k) \le 0$.

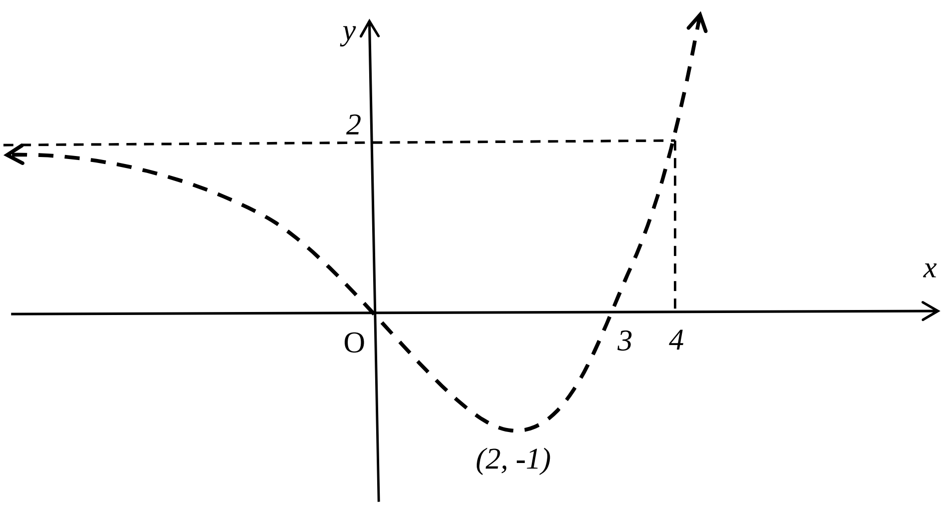
(iv) Hence show that $a_1 a_2 a_3 ....a_n \le \dfrac{1}{n^n}$. **2**

End of paper

This page is blank

Answer sheet for Question 12 Name ________________





CTHS 2013 AP4 Ext 2 Sol^n

1 C	2 B	3. C	4. C	5. A
6. B	7. A	8. A	9. D	10 D

Question 11

a) $z = 1+\sqrt{3}\,i$

(i) $|z| = \sqrt{1+3}$

$= 2$ [1]

$\arg z = \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$ [1]

(ii) $z = 2\,\text{cis}\frac{\pi}{3}$

$z^5 = 2^5\left(\text{cis}\frac{\pi}{3}\right)^5$

$= 2^5\,\text{cis}\frac{5\pi}{3}$

$= 32\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$ [1]

$= 32\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= \frac{1}{2}\times 32 - i\times 32\times\frac{\sqrt{3}}{2}$

$= 16 - 16\sqrt{3}\,i$ [1]

(iii) $z = 2\,\text{cis}\frac{\pi}{3}$

The 4 4th roots of z are $2^{\frac{1}{4}}\,\text{cis}\left(\frac{\frac{\pi}{3}+2k\pi}{4}\right)$ $k=0,1,2,3$ [1]

ie $2^{\frac{1}{4}}\,\text{cis}\frac{\pi}{12}$, $2^{\frac{1}{4}}\,\text{cis}\frac{7\pi}{12}$, $2^{\frac{1}{4}}\,\text{cis}\frac{13\pi}{12}$ and $2^{\frac{1}{4}}\,\text{cis}\frac{19\pi}{12}$ [1]

b) Let $15-8i = (a+bi)^2$ a, b are real nos.

$= a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = 15$ (1)

$ab = -4$ (2)

$b = -\frac{4}{a}$ (3)

Put (3) in (1) $a^2 - \frac{16}{a^2} = 15$

$a^4 - 15a^2 - 16 = 0$

$(a^2+1)(a^2-16) = 0$

$a^2 = -1$ (rejected) $a^2 = 16$

$\therefore a = \pm 4$

$b = \mp 1$ } [1]

$\therefore \sqrt{15-8i} = \pm(4-i)$ [1]

c) $\because PQRS$ is a square

$\therefore RP = SQ$ and

$RP \perp SQ$

ie $\overrightarrow{SQ} = i\times\overrightarrow{RP}$ [1]

$q - s = i(p-r)$ [1]

Hence $s + ip = q + ir$

Q11 (cont'd)

d) (i) $\frac{2x^3-9x^2+18x-9}{(1+x^2)(9+x^2)} = \frac{(ax+b)(9+x^2)+(cx+d)(1+x^2)}{(1+x^2)(9+x^2)}$

$\therefore (ax+b)(9+x^2)+(cx+d)(1+x^2) \equiv 2x^3-9x^2+18x-9$

$(a+c)x^3+(b+d)x^2+(9a+c)x+(9b+d)$
$\equiv 2x^3-9x^2+18x-9$

Equating coeff of x^3 $\quad a+c=2 \quad (1)$

$b+d=-9 \quad (2)$

$9a+c=18 \quad (3)$ [1]

$9b+d=-9 \quad (4)$

(3) − (1) $\quad 8a=16$

$a=2$ [1]

put into (1) $\quad c=0$

(4) − (2) $\quad 8b=0$

$b=0$ [1]

$\therefore \quad d=-9$

(ii) $\int_0^3 \frac{2x^3-9x^2+18x-9}{(1+x^2)(9+x^2)}dx = \int_0^3 \left(\frac{2x}{1+x^2} - \frac{9}{9+x^2}\right)dx$

$= \left[\ln(1+x^2)\right]_0^3 - 3\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$ [1]

$= \ln 10 - \frac{3\pi}{4}$ [1]

Question 12

a) (i) $P(x) = (x-a)^2Q(x)$ [1]

$P'(x) = 2(x-a)Q(x) + (x-a)^2Q'(x)$

$= (x-a)[2aQ(x) + (x-a)Q'(x)]$ [1]

$\therefore P'(a)=0$

Hence $P'(a) = P(a) = 0$

(ii) Let $P(x) = x^4-2x^3+x^2+12x+8$

$P'(x) = 4x^3-6x^2+2x+12$

by inspection $P(-1) = P'(-1) = 0$ [1]

$\therefore \quad x=-1$ is a double root of $P(x)=0$
by result of (i)

Let $P(x) \equiv (x+1)^2(x^2+ax+b)$

$x^4-2x^3+x^2+12x+8 \equiv (x+1)^2(x^2+ax+b)$

Equating constant terms $\quad 8 \equiv b$

put $x=1$ $\quad 20 = 2^2(1+a+8)$

$5 = 9+a$

$\therefore a=-4$

$\therefore P(x) = (x+1)^2(x^2-4x+8)$

Q12 (cont'd)

alt 1.

$$
\begin{array}{r}
x^2-4x+8 \\
x^2+2x+1 \overline{\big)\, x^4-2x^3+x^2+12x+8} \\
\underline{x^4+2x^3+x^2} \\
-4x^3 \qquad +12x \\
\underline{-4x^3-8x^2-4x} \\
8x^2+16x+8 \\
8x^2+16x+8
\end{array}
$$

alt 2.

-1	1	-2	1	12	8
		-1	3	-4	-8
-1	1	-3	4	8	0
		-1	4	-8	
	1	-4	8	0	

$\therefore P(x) = (x+1)^2(x^2-4x+8)$ [1]

$\therefore$ Roots are $-1, -1, 2\pm 2i$ [1]

(iii) Since $= 3+2i$ is a root,

$\therefore$ $= 3-2i$ must also be a root

$\therefore (x-3-2i)(x-3+2i) = [(x-3)-2i][(x-3)+2i]$ [1]

$= (x-3)^2+4$

$= x^2-6x+13$ is a factor

$x^4-5x^3+5x^2+25x-26 = 0$

$(x^2-6x+13)(x^2+x-2) = 0$ [1]

Roots of $x^2+x-2=0$ are $x = 1, -2$

$\therefore$ Roots are $1, -2, 3-2i, 3+2i$ [1]

(b) (i) $\dfrac{1}{\alpha\beta} = \dfrac{\gamma}{\alpha\beta\gamma}$

$= \dfrac{\gamma}{1/2}$

$= 2\gamma$

Let roots $y = 2x$

$x = \dfrac{y}{2}$ [1]

$2x^3 - x^2 + 3x - 1 = 0$ is transformed to

$2\left(\dfrac{y}{2}\right)^3 - \left(\dfrac{y}{2}\right)^2 + 3\left(\dfrac{y}{2}\right) - 1 = 0$

$\dfrac{y^3}{4} - \dfrac{y^2}{4} + \dfrac{3y}{2} - 1 = 0$

$y^3 - y^2 + 6y - 4 = 0$

$\therefore$ Required equation is

$x^3 - x^2 + 6x - 4 = 0$ [1]

(ii) Let $y = x^2$, ie $x = \sqrt{y}$

$2x^3 - x^2 + 3x - 1 = 0$ is transformed to

$2y\sqrt{y} - y + 3\sqrt{y} - 1 = 0$

$\sqrt{y}(2y+3) = y+1$ [1]

$y(2y+3)^2 = (y+1)^2$

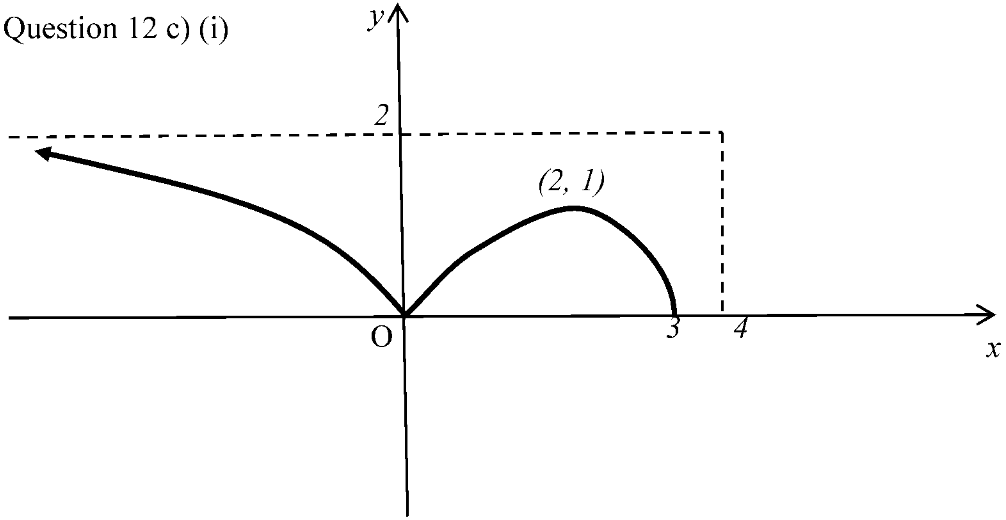
$y(4y^2+12y+9) = y^2+2y+1$

$4y^3+12y^2+9y = y^2+2y+1$

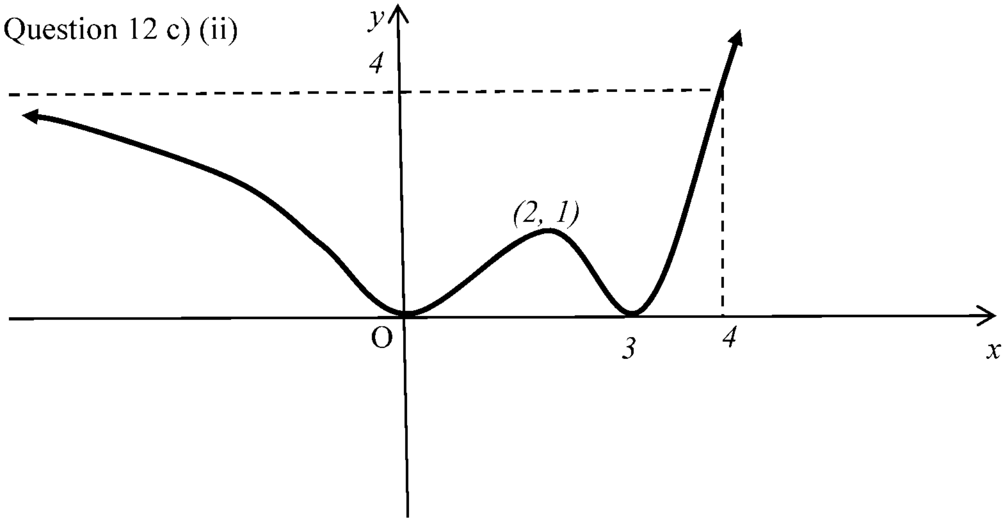
$4y^3+11y^2+7y-1 = 0$

New eq^n is $4x^3+11x^2+7x-1=0$ [1]

Question 12 c) (i)



Question 12 c) (ii)



Question 13

a) (i) Since $\angle AOB = \frac{\pi}{4}$ $(\angle OAB = \frac{\pi}{2},\ OA = AB)$

$OB = \sqrt{2}\, OA$

$\therefore \overrightarrow{OB} = \sqrt{2} \operatorname{cis}\frac{\pi}{4} \times \overrightarrow{OA}$ [1]

$= \sqrt{2} \operatorname{cis}\frac{\pi}{4} \times \alpha$

$\therefore = \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)\alpha$

(ii) Similarly $\overrightarrow{OC} = \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right) \times \overrightarrow{OB}$

$= \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)\left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)\alpha$

$= \left(2 \operatorname{cis}\frac{\pi}{2}\right)\alpha$

$= 2i\alpha$ [1]

$\overrightarrow{OD} = \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right) \times \overrightarrow{OC}$

$= \left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)(2i\alpha)$

$\therefore \beta = \left(2\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right) i\alpha$ [1]

(iii) $\beta^4 = \left(2\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)^4 (i\alpha)^4$

$= (64 \operatorname{cis}\pi)\alpha^4$ [1]

$= -64\alpha^4$

$\therefore 64\alpha^4 + \beta^4 = 0$

Q13 (cont'd)

b) Let $u = x+1$

$\therefore x = u-1$

$dx = du$

when $x=3, \quad u=4$

$x=0, \quad u=1$ [1]

$\therefore \int_0^3 x\sqrt{x+1}\,dx = \int_1^4 (u-1)\sqrt{u}\,du$

$= \int_1^4 (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$ [1]

$= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$

$= \frac{2}{5}(32-1) - \frac{2}{3}(8-1)$

$= 7\frac{11}{15}$ or $\frac{116}{15}$. [1]

c) (i) $(p-\frac{1}{p})^2 \geq 0$

$p^2 - 2 + \frac{1}{p^2} \geq 0$ [1]

$\therefore p^2 + \frac{1}{p^2} \geq 2$

(ii) (α) AB : $x + p^2 y = 2cp$

At A $\quad y=0 \quad x = 2cp \quad \therefore$ A is $(2cp, 0)$

At B, $\quad x=0 \quad p^2 y = 2cp$

$y = \frac{2c}{p} \quad$ B is $(0, \frac{2c}{p})$ [1]

$\therefore AB = \sqrt{(2cp)^2 + (\frac{2c}{p})^2}$ [1]

$= 2c\sqrt{p^2 + \frac{1}{p^2}}$

(β) Let Q be $(cq, \frac{c}{q})$

$\therefore \frac{cp}{q} - c = p^3(cq - cp)$

$\frac{p}{q} - 1 = p^3(q-p)$

$p - q = p^3 q(q-p)$

$\therefore q = -\frac{1}{p^3}$ [1]

$\therefore$ Q is $(-\frac{c}{p^3}, -cp^3)$ [1]

(γ) $PQ = \sqrt{(cp + \frac{c}{p^3})^2 + (\frac{c}{p} + cp^3)^2}$

Q13 (cont'd)

$$PQ = \sqrt{c p^2 + \frac{2c^2}{p^2} + \frac{c^2}{p^6} + \frac{c^2}{p^2} + 2c^2p^2 + c^2p^6}$$

$$= c\sqrt{p^6 + 3p^2 + \frac{3}{p^2} + \frac{1}{p^6}}$$

$$= c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3} \quad \boxed{1}$$

$$\therefore \triangle ABQ = \frac{1}{2} PQ \times AB$$

$$= \frac{1}{2} c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3} \times 2c\sqrt{p^2 + \frac{1}{p^2}} \quad \boxed{1}$$

$$= c^2\left(p^2 + \frac{1}{p^2}\right)^2$$

(δ) From (c) $p^2 + \frac{1}{p^2} \geq 2$

$$\therefore \triangle ABQ = c^2\left(p^2 + \frac{1}{p^2}\right)^2$$

$$\geq c^2(2^2)$$

$$= 4c^2$$

$\therefore$ min. area of $\triangle ABQ$ is $4c^2$. $\boxed{1}$

Question 14

a) (i) $\sqrt{x^2 + (y-3)^2} + \sqrt{x^2 + (y+3)^2} = 10$ $\boxed{1}$

$$\sqrt{x^2 + (y-3)^2} = 10 - \sqrt{x^2 + (y+3)^2} \quad \boxed{1}$$

$$x^2 + (y-3)^2 = 100 - 20\sqrt{x^2 + (y+3)^2} + x^2 + (y+3)^2$$

$$20\sqrt{x^2 + (y+3)^2} = 100 + (y+3)^2 - (y-3)^2$$

$$= 100 + 12y$$

$$5\sqrt{x^2 + (y+3)^2} = 25 + 3y \quad \boxed{1}$$

$$25\left[x^2 + (y+3)^2\right] = (25 + 3y)^2$$

$$25x^2 + 25y^2 + 150y + 225 = 625 + 150y + 9y^2$$

$$25x^2 + 16y^2 = 400 \quad \boxed{1}$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

ii) when $y = 4$

$$\frac{x^2}{16} + \frac{16}{25} = 1$$

$$x^2 = 16\left(1 - \frac{16}{25}\right)$$

$$= 16 \times \frac{9}{25}$$

$\therefore x = \frac{12}{5}$ (in 1st quadrant) $\boxed{1}$

Q 14 (cont'd)

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

$$\frac{x}{8}+\frac{2y}{25}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{25x}{16y}$$

$\therefore$ at $\left(\frac{12}{5}, 4\right)$

$$\frac{dy}{dx}=-\frac{25}{16}\times\frac{12/5}{4}$$

$$=-\frac{15}{16} \qquad \boxed{1}$$

$\therefore$ Equation of tangent is

$$y-4=-\frac{15}{16}\left(x-\frac{12}{5}\right)$$

$$16y-64=-15x+36$$

ie $15x+16y-100=0$ $\boxed{1}$

(b) $t=\tan\frac{x}{2}$

$$dx=\frac{2\,dt}{1+t^2}$$

when $x=\frac{\pi}{3}$, $t=\frac{1}{\sqrt{3}}$

$x=0$, $t=0$ $\Bigg\}$ $\boxed{1}$

$$\therefore \int_0^{\frac{\pi}{3}}\frac{1}{1+\cos x-\sin x}\,dx$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{(1+t^2)\left[1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}\right]}\,dt$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{1+t^2+1-t^2-2t}\,dt$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{1-t} \qquad \boxed{1}$$

$$=-\left[\ln(1-t)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$=-\ln\left(1-\frac{1}{\sqrt{3}}\right)$$

$$=-\ln\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$=\ln\frac{\sqrt{3}}{\sqrt{3}-1}$$

$$=\ln\frac{3+\sqrt{3}}{2} \qquad \boxed{1}$$

c) $P(x)=ax^3+3x+b$

$P'(x)=3ax^2+3$

At turning pts $P'(x)=0$

Q14 (cont'd)

$\therefore\ 3ax^2+3=0$

$x^2=-\frac{1}{a}$ (1)

Since $y=P(x)$ has 2 turning points,

$\therefore$ equation (1) must have 2 real roots

hence $x^2=-\frac{1}{a}>0$ [1]

ie $a<0$

(ii) product of roots

$(m+in)(m-in)\frac{1}{a}=-\frac{b}{a}$ [1]

$(m^2+n^2)=-b$

ie $b=-(m^2+n^2)$ [1]

<0 $\because m,n$ are real

(iii) Sum of roots taken 2 at a time:

$[(m+in)+(m-in)]\cdot\frac{1}{a}+(m+in)(m-in)=\frac{3}{a}$

$\frac{2m}{a}+m^2+n^2=\frac{3}{a}$ [1]

$\therefore \frac{2m}{a}=\frac{3}{a}-(m^2+n^2)$

$<\frac{3}{a}$ $\because m^2>0, n^2>0$ [1]

$\therefore m>\frac{3}{2}$ $\because a<0$

Question 15

a) The width of the rectangular crossection

$=\sqrt{a^2(1-\frac{y^2}{b^2})}-\sqrt{b^2-y^2}$

$=\frac{a}{b}\sqrt{b^2-y^2}-\sqrt{b^2-y^2}$

$=(\frac{a}{b}-1)\sqrt{b^2-y^2}$

$=\frac{a-b}{b}\sqrt{b^2-y^2}$ [1]

$\therefore$ Area of the rectangular cross-section

$=\frac{a-b}{b}\sqrt{b^2-y^2}\cdot b$

$=(a-b)\sqrt{b^2-y^2}$

Volume of the slice

$\delta V=(a-b)\sqrt{b^2-y^2}\,\delta y$ [1]

Hence volume of the solid

$V=\lim_{\delta y\to 0}\sum \delta V$

$=(a-b)\int_0^b\sqrt{b^2-y^2}\,dy$

$=(a-b)\frac{\pi b^2}{4}$

$=\frac{\pi(a-b)b^2}{4}$ unit^3 [1]

Q15 (cont'd)

b) (i) By Newton's 2nd law of motion,

$$F = ma$$

$$F = -(v+v^3) \quad \& \quad m = 1$$ [1]

$$\therefore a = -(v+v^3)$$

(ii) Since $a = v\frac{dv}{dx}$

$$\therefore v\frac{dv}{dx} = -(v+v^3)$$ [1]

$$= -v(1+v^2)$$

$$\frac{dv}{dx} = -(1+v^2)$$

$$-\int_{\sqrt{3}}^{v} \frac{dv}{1+v^2} = \int_0^x dx$$ [1]

$$-\left[\tan^{-1} v\right]_{\sqrt{3}}^{v} = x$$

$$\therefore x = \tan^{-1}\sqrt{3} - \tan^{-1} v$$ [1]

$$\tan x = \tan(\tan^{-1}\sqrt{3} - \tan^{-1} v)$$

$$= \frac{\sqrt{3} - v}{1 + \sqrt{3}\,v}$$ [1]

$$\therefore x = \tan^{-1}\left(\frac{\sqrt{3} - v}{1 + \sqrt{3}\,v}\right)$$

(iii) from (i) $a = \frac{dv}{dt} = -v(1+v^2)$

$$\therefore \int_{\sqrt{3}}^{V} \frac{dv}{v(1+v^2)} = -\int_0^t dt$$ [1]

$$\int_{\sqrt{3}}^{V} \left(\frac{1}{v} - \frac{v}{1+V^2}\right) dv = -\left[t\right]_0^t$$ [1]

$$\therefore t = \left[\ln v - \frac{1}{2}\ln(1+v^2)\right]_V^{\sqrt{3}}$$

$$= \frac{1}{2}\left[\ln v^2 - \ln(1+v^2)\right]_V^{\sqrt{3}}$$ [1]

$$= \frac{1}{2}\left[\ln \frac{v^2}{1+v^2}\right]_V^{\sqrt{3}}$$

$$= \frac{1}{2}\left[\ln\frac{3}{4} - \ln\frac{V^2}{1+V^2}\right]$$

$$= \frac{1}{2}\ln\left[\frac{3(1+V^2)}{4V^2}\right]$$ [1]

(iv) $$2t = \ln\frac{3(1+V^2)}{4V^2}$$

$$e^{2t} = \frac{3(1+V^2)}{4V^2}$$ [1]

Q15 (cont'd)

$$4V^2e^{2t} = 3 + 3V^2$$

$$V^2(4e^{2t} - 3) = 3$$

$$\therefore \quad V^2 = \frac{3}{4e^{2t} - 3} \qquad \boxed{1}$$

(v) $\therefore V \to 0$ as $t \to \infty$

From (β) $\quad x = \tan^{-1}\left(\frac{\sqrt{3} - v}{1 + v\sqrt{3}}\right)$

$\therefore$ as $v \to 0$

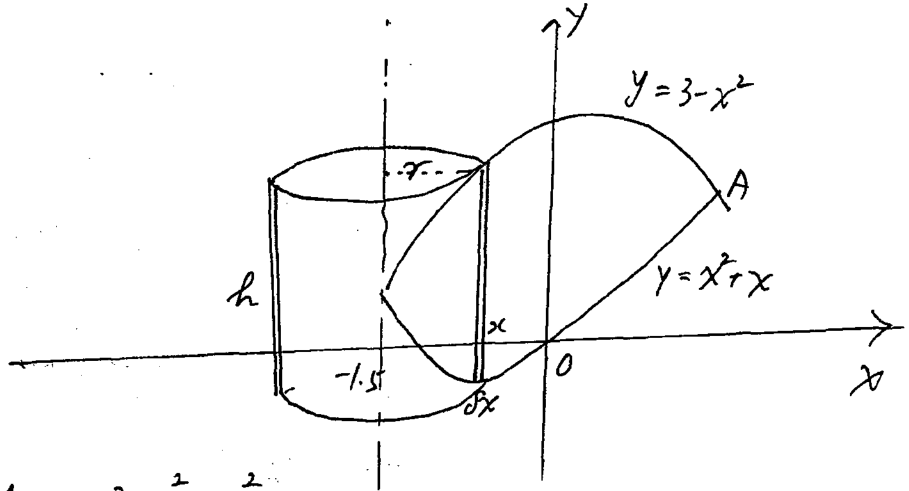
$$x \to \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$= \frac{\pi}{3}$$

$\therefore$ Limiting position is $\frac{\pi}{3}$ $\boxed{1}$

Question 16

a)



At A $\quad 3 - x^2 = x^2 + x$

$$2x^2 + x - 3 = 0$$

$$(x-1)(2x+3) = 0$$

$x = 1 \qquad x = -\frac{2}{3}$ (rejected) $\boxed{1}$

$$h = (3-x^2) - (x^2+x)$$

$$= 3 - x - 2x^2 \qquad \boxed{1}$$

and $r = x - (-1.5) = x + 1.5$

$\therefore$ Volume of the cylindrical shell

$$\delta V = 2\pi r h \, \delta x$$

$$= 2\pi(x+1.5)(3-x-2x^2)\,\delta x$$

$$\therefore V = \lim_{\delta x \to 0} \sum \delta V$$

$$= 2\pi \int_{-1.5}^{1} (x+1.5)(3-x-2x^2)\,dx$$

Q16 (cont'd)

$$\bar{V} = 2\pi\int_{-1.5}^{1}(3x - x^2 - 2x^3 + \frac{9}{2} - \frac{3x}{2} - 3x^2)\,dx$$

$$= 2\pi\int_{-1.5}^{1}(\frac{9}{2} + \frac{3x}{2} - 4x^2 - 2x^3)\,dx$$

$$= 2\pi\left[\frac{9x}{2} + \frac{3x^2}{4} - \frac{4x^3}{3} - \frac{x^4}{2}\right]_{-1.5}^{1} \quad \boxed{1}$$

$$= 2\pi\left[\left(\frac{9}{2} + \frac{3}{4} - \frac{4}{3} - \frac{1}{2}\right) - \left(-\frac{9}{2}\times\frac{3}{2} + \frac{3}{4}\left(\frac{3}{2}\right)^2 - \frac{4}{3}\left(-\frac{3}{2}\right)^3 - \frac{1}{2}\left(\frac{3}{2}\right)^4\right)\right]$$

$$= \frac{625\pi}{48} \text{ unit}^3 \quad \boxed{1}$$

<u>alt</u> Let $u = x + 1.5 \quad \therefore x = u - 1.5$

when $x = 1 \quad u = 2.5$

$x = -1.5 \quad u = 0$

$dx = du$

$$\bar{V} = 2\pi\int_{-1.5}^{1}(x+1.5)(3 - x - 2x^2)\,dx$$

$$= 2\pi\int_{0}^{2.5}u\left[3 - (u - \frac{3}{2}) - 2(u - \frac{3}{2})^2\right]du$$

$$= 2\pi\int_{0}^{2.5}u\left[3 - u + \frac{3}{2} - 2(u^2 - 3u + \frac{9}{4})\right]du$$

$$= 2\pi\int_{0}^{2.5}u(\frac{9}{2} - u - 2u^2 + 6u - \frac{9}{2})\,du$$

$$= 2\pi\int_{0}^{2.5}u(5u - 2u^2)\,du$$

$$= 2\pi\int_{0}^{2.5}5u^2 - 2u^3\,du$$

$$= 2\pi\left[\frac{5u^3}{3} - \frac{u^4}{2}\right]_{0}^{2.5}$$

$$= 2\pi\left[\frac{5}{3}\left(\frac{5}{2}\right)^3 - \frac{1}{2}\left(\frac{5}{2}\right)^4\right]$$

$$= \frac{625\pi}{48} \text{ unit}^3$$

(b) (i) $\cos 3\theta = \cos(\theta + 2\theta)$

$$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$$

$$= \cos\theta(2\cos^2\theta - 1) - \sin\theta(2\sin\theta\cos\theta) \quad \boxed{1}$$

$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

(ii) $\cos 3\theta + \cos\theta + 4\cos^3\theta$

$$= 6\cos^3\theta - 2\sin^2\theta\cos\theta$$

$$= 2\cos\theta(3\cos^2\theta - \sin^2\theta) \quad \boxed{1}$$

$$= 8\cos\theta(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta)$$

$$= 8\cos\theta(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta)(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta) \quad \boxed{1}$$

Q16 (cont'd)

$= 8\cos\theta\left(\cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}\right)\left(\cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6}\right)$ [1]

$= 8\cos\theta\cos\left(\theta+\frac{\pi}{6}\right)\cos\left(\theta-\frac{\pi}{6}\right)$

c) (i) $y = \ln x$

$y' = \frac{1}{x}$

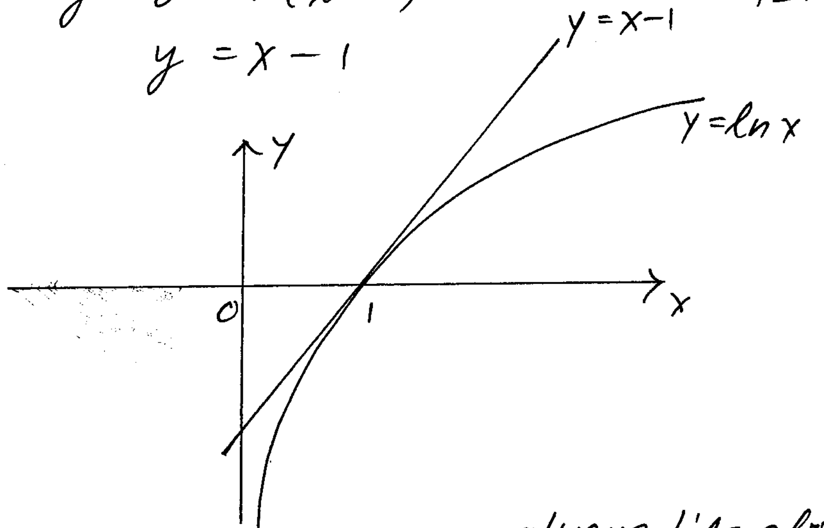
at $x=1$, $y'=1$ $y=0$

$\therefore$ Eq^n of tangent at $x=1$ is

$y - 0 = 1(x-1)$ [1]

$y = x-1$

(ii)



The graph of $y=x-1$ always lies above [1]
the graph of $y=\ln x$ which is concave down.

$\therefore$ $x-1 \geq \ln x$

or $\ln x \leq x-1$ [1]

(iii) $\sum_{k=1}^{n} \log_e(na_k) \leq \sum_{k=1}^{n}(na_k - 1)$ (from (ii)) [1]

$= n\sum_{k=1}^{n} a_k - n$

$= n - n$ $\therefore a_1+a_2+\cdots+a_n = 1$ [1]

$= 0$

$\therefore \sum_{k=1}^{n}\log_e(na_k) \leq 0$

(iv) $\sum_{k=1}^{n} \ln(na_k)$

$= \ln na_1 + \ln na_2 + \cdots + \ln na_n$

$= \ln(na_1)(na_2)\cdots(na_n)$ [1]

$= \ln n^n(a_1a_2\cdots a_n)$

from (iii) $\ln n^n(a_1a_2\cdots a_n) \leq 0$

ie $n^n(a_1a_2\cdots a_n) \leq 1$ [1]

$a_1a_2\cdots a_n \leq \frac{1}{n^n}$